Force Between Multiple Charges

Objectives

After going through this lesson, the learners will be able to:

- express the principle of superposition
- Evaluate interaction between the multiple charges
- Define continuous charge distribution
- Apply Coulomb's law to solve numerical problems

Content Outline

- Unit syllabus
- Module-wise distribution of unit syllabus
- Words you must know
- Introduction
- Forces between multiple charges- principle of superposition
- Numerical problems
- Continuous distribution of charges
- Summary

Unit Syllabus

UNIT 1: Electrostatics:

Chapter-1: Electric Charges and Fields

Electric Charges; Conservation of charge, Coulomb's law-force between two point charges, forces between multiple charges; superposition principle and continuous charge distributions.

Electric field; electric field due to a point charge, electric field lines, electric dipole, electric field due to a dipole, torque on a dipole in uniform electric field.

Electric flux, statement of Gauss's theorem and its applications to find field due to infinitely long straight wire, uniformly charged infinite plane sheet and uniformly charged thin spherical shell (field inside and outside).

Chapter-2: Electrostatic Potential and Capacitance

Electric potential, potential difference, electric potential due to a point charge, a dipole and system of charges; equipotential surfaces, electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field.

Conductors and insulators, free charges and bound charges inside a conductor. Dielectrics and electric polarization, capacitors and capacitance, combination of capacitors in series and in parallel, capacitance of a parallel plate capacitor with and without dielectric medium between the plates, energy stored in a capacitor.

Module-Wise Distribution Of Unit Syllabus

The above unit is divided into 11 modules for better understanding.

Module 1	• Electric charge • Properties of charge • Coulomb's law • Characteristics of coulomb force • Effect of intervening medium on the coulomb force • Examples
Module 2	• Forces between multiple charges • Principle of superposition • Continuous distribution of charges • Numerical
Module 3	• Electric field E • Importance of field E and ways of describing field • Point charges superposition of electric field • Examples
Module 4	• Electric dipole • Electric field of a dipole • Charges in external field • Dipole in external field Uniform and non-uniform
Module 5	• Electric flux , • Flux density • Gauss theorem • Application of gauss theorem to find electric field • For a distribution of charges

	• Numerical
Module 6	• Application of gauss theorem Field due to field infinitely long straight wire • Uniformly charged infinite plane • Uniformly charged thin spherical shell (field inside and outside) • Graphs
Module 7	• Electric potential, • Potential difference, • Electric potential due to a point charge, a dipole and system of charges; • Equipotential surfaces, • Electrical potential energy of a system of two point charges and of electric dipole in an electrostatic field. • Numerical
Module 8	• Conductors and insulators, • Free charges and bound charges inside a conductor. • Dielectrics and electric polarization
Module 9	• Capacitors and Capacitance, • Combination of capacitors in series and in parallel • Redistribution of charges , common potential • numerical
Module 10	• Capacitance of a parallel plate capacitor with and without dielectric medium between the plates • Energy stored in a capacitor
Module 11	• Typical problems on capacitors

Words You Must Know

Let us recollect the words we have been using in our study of this physics course.

- **Electric Charge:** Electric charge is an intrinsic characteristic of many of the fundamental particles of matter that gives rise to all electric and magnetic forces and interactions.

- **Conductors:** Some substances readily allow passage of electricity through them, others do not. Those which allow electricity to pass through them easily are called *conductors*. They have electric charges (electrons) that are comparatively free to move inside the material. Metals, human and animal bodies and earth are all conductors of electricity.
- **Insulators:** Most of the non-metals, like glass, porcelain, plastic, nylon, wood, offer high opposition to the passage of electricity through them. They are called *insulators.*
- **Point Charge:** When the linear size of charged bodies is much smaller than the distance separating them, the size may be ignored and the charge bodies can then be treated as *point charges*.
- **Conduction:** Transfer of electrons from one body to another, it also refers to flow of charged electrons in metals and ions in electrolytes and gases.
- **Induction:** The temporary separation of charges in a body due to a charged body in the vicinity. The effect lasts as long as the charged body is held close to the body in which induction is taking place.
- **Quantisation of charges:** Charge exists as an integral multiple of basic electronic charge. Charge on an electron is $1.6\times10^{-19}\ C$.
- **Electroscope:** A device to detect charge.
- **Coulomb:** S.I unit of charge defined in terms of 1 ampere current flowing in a wire to be due to 1 coulomb of charge flowing in 1 s.

  1 coulomb = collective charge of 6×10^{18} electrons
- **Conservation of charge:** Charge can neither be created nor be destroyed in an isolated system, it (electrons) only transfers from one body to another.
- **Coulomb's Force:** It is the electrostatic force of interaction between the two point charges
- **Vector form of coulomb's law:** A mathematical expression based on coulomb's law to show the magnitude as well as direction of mutual electrostatic force between two or more charges.
- **Laws of vector addition**

  Triangle law of vector addition: If two vectors are represented by two sides of a triangle in order, then the third side represents the resultant of the two vectors.

Parallelogram law of vector addition: If two vectors are represented in magnitude and direction by adjacent sides of a parallelogram then the resultant of the vectors is given by the diagonal passing through their common point.

Also resultant of vectors P and Q acting at angle of θ is given by

$$R = \sqrt{P^2 + Q^2 + 2PQcos\theta}$$

Polygon law of vector addition: Multiple vectors may be added by placing them in order of a multi sided polygon, the resultant is given by the closing side taken in opposite order.

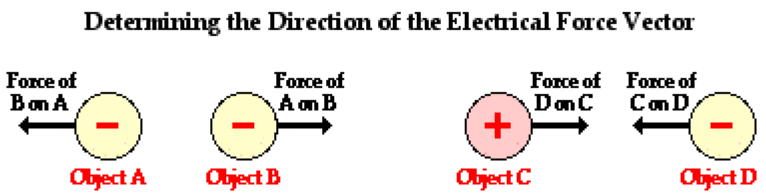
Introduction

As we have already studied in the 1st module about interaction between the two point charges. Let's recall some important points about this interaction.

The electrical force, like all forces, is typically expressed using the unit Newton. Being a force, the strength of the electrical interaction is a vector quantity that has both magnitude and direction.

The direction of the electrical force is dependent upon whether the interacting bodies carry like charges or unlike charges and upon their position and orientation.

By knowing the type of charge on the two objects, the direction of the force on either one of them can be determined with a little reasoning. In the diagram below, objects A and B possess a charge causing them to repel each other. Thus, the force on object A is directed leftward (away from B) and the force on object B is directed rightward (away from A). On the other hand, objects C and D have opposite charges causing them to attract each other. Thus, the force on object C is directed rightward (towards object D) and the force on object D is directed leftward (towards object C).



Coulomb's Law Equation

The quantitative expression for the electric force is known as Coulomb's law.

Coulomb's law states that the electrical force between two charged objects is directly proportional to the product of the quantity of charge on the objects and inversely proportional to the square of the distance between the centers of the two objects.

In equation form, Coulomb's law can be stated as

$$F = K\frac{q_1 \times q_2}{r^2}$$

Now in this module 2 we will study about the interaction between more than two point charges in space and how coulomb's law is modified for continuous charge distributions.

Forces between Multiple Charges – Principle of Superposition of Charges

The mutual electric force between two charges is given by Coulomb's law.

Now we will learn to calculate the net force on a charge due to several charges around?

The charges may be arranged along a straight line or distributed in 2 dimensions or in 3 dimensional spaces. In each of the cases we must bear in mind the position vectors of the influencing charges with respect to the charge on which we wish to find the net force.

Consider a system of *n* stationary charges $q_1,\ q_2,\ q_3,\ ...,q_n$ in a vacuum. What is the force on q_1 due to $q_1,\ q_2,\ q_3,\ ...,q_n$?

Is Coulomb's law enough to answer this question?

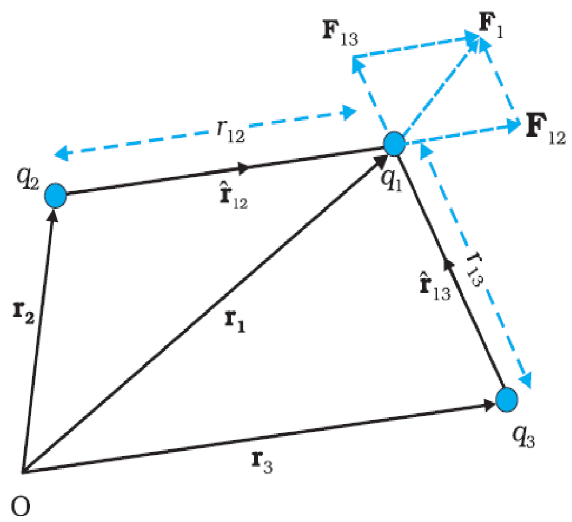
Recall that forces producing mechanical change of position, add according to the laws of vector addition.

Is the same true for forces of electrostatic origin?

Experimentally, it can be verified that force on any charge due to a number of other charges is the vector sum of all the forces on that charge due to the other charges, taken one at a time. The individual forces are unaffected due to the presence of other charges. This is termed as the principle of superposition.

To understand the concept better, consider a system of three charges q_1, q_2 and q_3, as shown in Fig.

The force on one charge, say q_1, due to two other charges q_2, q_3 can therefore be obtained by performing a vector addition of the forces due to each one of these charges.



Thus, if the force on q_1 due to q_2 is denoted by F_{12} ,

Then F_{12} is given by –

$$F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2_{12}}\hat{r}_{12}$$

(Even though other charges q_3 is present in the space)

In the same way, the force on q_1 due to q_3 can be calculated.

Thus the total force F_1 on q_1 due to the two charges q_2 and q_3 is given as:

$$F_1 = F_{12} + F_{13} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2_{12}}\hat{r}_{12} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r^2_{13}}\hat{r}_{13}$$

The above calculation of force can be generalized to a system of charges more than three.

The principle of superposition says that in a system of charges $q_1, q_2, q_3, ..., q_n$ the force on q_1 due to q_2 is the same as given by Coulomb's law, i.e., it is unaffected by the presence of the other charges $q_3, q_4, ..., q_n$. Then:

$$F_1 = F_{12} + F_{13} + ... + F_{1n} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r^2_{12}}\hat{r}_{12} + \frac{q_1 q_3}{r^2_{13}}\hat{r}_{13} + ... + \frac{q_1 q_n}{r^2_{1n}}\hat{r}_{1n}\right] = \frac{q_1}{4\pi\varepsilon_0}\sum_{i=2}^{n}\frac{q_i}{r^2_{1i}}\hat{r}_{1i}$$

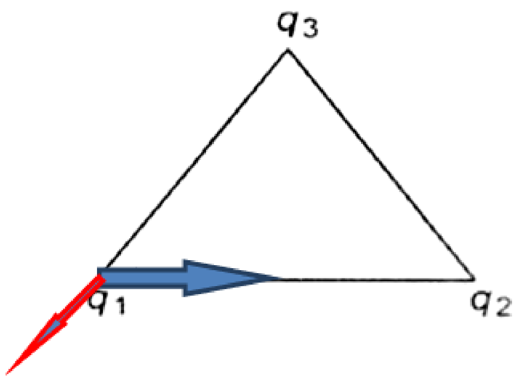
Total force F_1 on the charge q_1, due to all other charges, is then given by the vector sum of the forces F_{12}, F_{13} , ...F_{1n}; which is obtained by the parallelogram law of addition of vectors or by resolution of vectors (i.e. by finding components of force).
This will become clearer with the help of some examples.

Some Numerical Examples

It is easier to understand the principle of superposition if we give values to charges and distances and specify the distribution in space

Example:

Three charges $q_1 = 1mC$ (milli coulomb), $q_2 = -2mC$ and $q_3 = 3mC$ are placed on the vertices of an equilateral triangle of 1.0 m. Find the net electric force acting on charge q_1.



Solution:

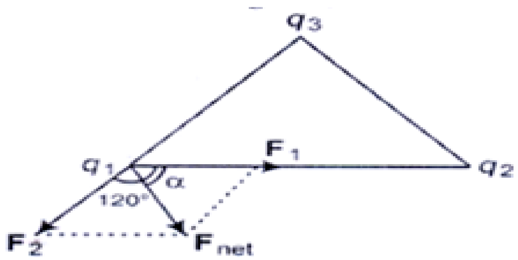
After drawing a schematic diagram

- Figure out the forces
- Charge q_2 will attract charge q_1 (along the line joining them) and
- Charge q_3 will repel charge q_1.
- Therefore, two forces will act on q_1, one due to q_2 and another due to q_3.

Since the force is a vector quantity both of these forces (say F_1 blue arrow and F_2 red arrow) will be added by the vector method. Following are two methods of their addition.

Method 1

Using parallelogram law of addition of vectors



$$|F_1| = F_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = magnitude\ of\ force\ between\ q_1 and\ q_2$$

$$= \frac{(9.0\times10^9)(1.0\times10^{-3})(2.0\times10^{-3})}{(1.0)^2} = 1.8\times10^4 N$$

Similarly,

$$|F_2| = F_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r^2} = magnitude\ of\ force\ between\ q_1 and\ q_3$$

$$= \frac{(9.0\times10^9)(1.0\times10^{-3})(3.0\times10^{-3})}{(1.0)^2} = 2.7\times10^4 N$$

Now, $$|F_{net}| = \sqrt{F_1^{\ 2} + F_2^{\ 2} + F_1 F_2 cos120^0}$$

$$= \sqrt{(1.8\times10^2)^2 + (2.7\times10^2)^2 + (2.7\times10^2)(1.8\times10^2)\left(\frac{-1}{2}\right)} = 2.85\times10^4$$

And

$$tan\alpha = \frac{F_2 sin120^0}{F_1 + F_2\, cos120^0}$$

$$= \frac{(2.7\times10^4)(0.87)}{(1.8\times10^4)+(2.7\times10^4)\left(\frac{-1}{2}\right)} = 5.22$$

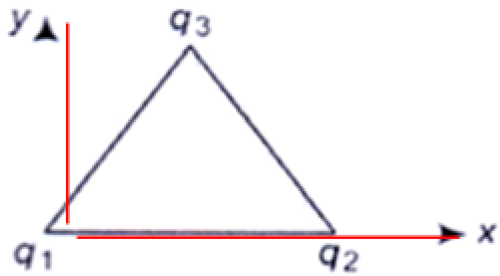
$$\alpha = 79.2^0$$

Thus, the net force on charge q_1 is 2.85×10^4 at an angle $\alpha = 79.2^0$ with a line joining q_1 and q_2 as shown in figure.

Method 2

In this method let us assume coordinate axes with q_1 at origin as shown in figure.

The coordinates of q_1, q_2 and q_3 in this coordinate system are (0, 0, 0), (1, 0, 0) and (0.5, 0.87, 0) respectively. We find components of force.



$$F_1 = force\ on\ q_1\ due\ to\ charge\ q_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|r_1 - r_2|^3}(r_1 - r_2)$$

$$F_1 = \frac{(9.0\times10^9)(1.0\times10^{-3})(-2.0\times10^{-3})\times[(0-1)\hat{i}+(0-0)\hat{j}+(0-0)\hat{k}]}{(1.0)^3} = (1.8\times10^2\hat{i})\ N$$

$$F_2 = force\ on\ q_1\ due\ to\ charge\ q_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{|r_1 - r_3|^3}(r_1 - r_3)$$

$$F_2 = \frac{(9.0\times10^9)(1.0\times10^{-3})(3.0\times10^{-3})\times[(0-0.5)\hat{i}+(0-0.87)\hat{j}+(0-0)\hat{k}]}{(1.0)^3} = (-1.35\hat{i} - 2.349\hat{j})\times10^2\ N$$

Therefore, net force on q_1 is:

$F = F_1 + F_2 = (0.45\hat{i} - 2.349\hat{j})\times10^2 N$

Note: Once you write a vector in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$, there is no need of writing the magnitude and direction of the vector separately.

Example:

Two identical balls each having a density ρ are suspended from a common point by two insulating strings of equal length. Both the balls have equal mass and charge. In equilibrium each string makes an angle θ with vertical.

Now, both the balls are immersed in a liquid. As a result the angle θ does not change. The density of the liquid is σ. Find the dielectric constant of the liquid.

Solution:

Each ball is in equilibrium under the following three forces:

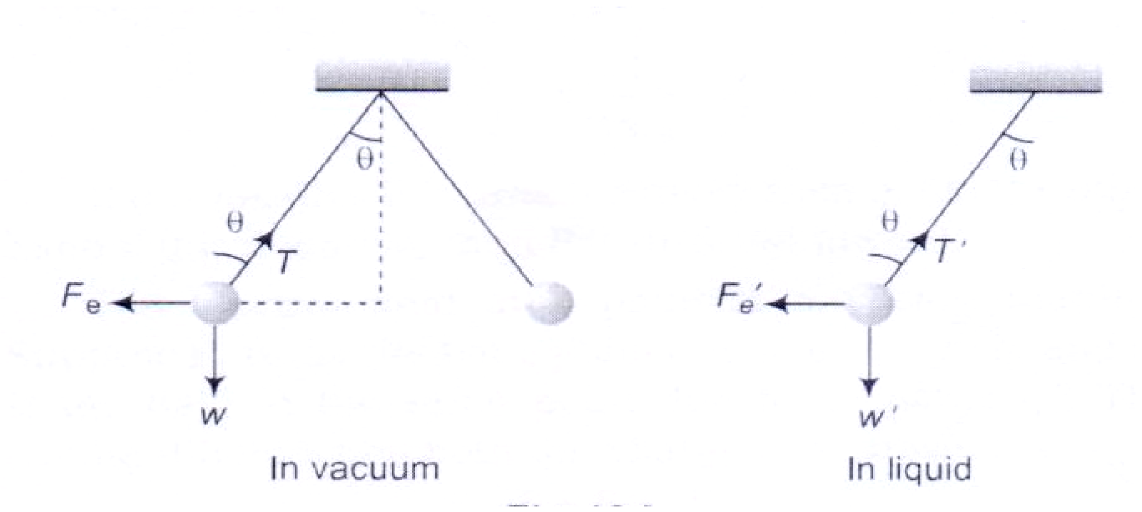
- Tension
- Electric force and
- Weight, as shown in the figure

So, Lami's theorem (for concurrent forces) can be applied.

Lami's theorem is an equation relating the magnitudes of three coplanar, concurrent and non-collinear forces, which keeps an object in static equilibrium, with the angles directly opposite to the corresponding forces. According to the theorem,

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$

Where A, B and C are the magnitudes of three coplanar, concurrent and non-collinear forces, which keep the object in static equilibrium, and α, β and γ are the angles directly opposite to the forces A, B and C respectively.



In the liquid, $F_e' = \frac{F_e}{k}$ where k is dielectric constant of liquid

And $w' = w - upthrust$

Applying Lami's theorem in vacuum:

$$\frac{w}{sin(90^0+\theta)} = \frac{F_e}{sin(180^0-\theta)}$$

Or

$$\frac{w}{cos\theta} = \frac{F_e}{sin\theta}\ldots\ldots\ldots \quad (1)$$

Similarly in liquid,

$$\frac{w'}{cos\theta} = \frac{F_e'}{sin\theta}\ldots\ldots\ldots \quad (2)$$

By dividing eq. (1) by eq. (2), we get:

$$\frac{w}{w'} = \frac{F_e}{F_e'}$$

Or

$$K = \frac{w}{w-upthrust} \qquad (\text{as } k = \frac{F_e}{F_e'})$$

$$K = \frac{V\rho g}{V\rho g - V\sigma g} \qquad (V = \text{volume of ball})$$

Or

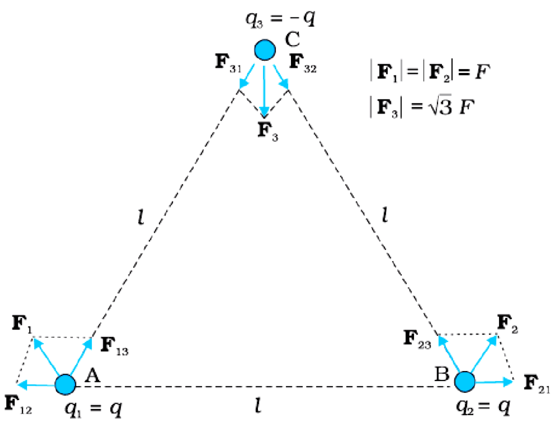
$$K = \frac{\rho}{\rho-\sigma}$$

Note: In the liquid F_e and w have been changed. Therefore, T will also change.

Example:

Consider the charges q, q, and $-q$ placed at the vertices of an equilateral triangle. What is the force on each charge?

Solution:



The forces acting on charge q at A due to charges q at B and $-q$ at C are F_{12} along BA and F_{13} along AC respectively, as shown in Fig. By the parallelogram law, the total force F_1 on the charge q at A is given by:

$F_1 = F_1 \hat{r}_1$ where $\hat{r}_1$ is a unit vector along BC.

The force of attraction or repulsion for each pair of charges has the same magnitude given by:

$$F = \frac{q^2}{4\pi\varepsilon_0 l^2}$$

The total force F_2 on charge q at B is thus $F_2 = F\hat{r}_2$, where $\hat{r}_2$ is a unit vector along AC.

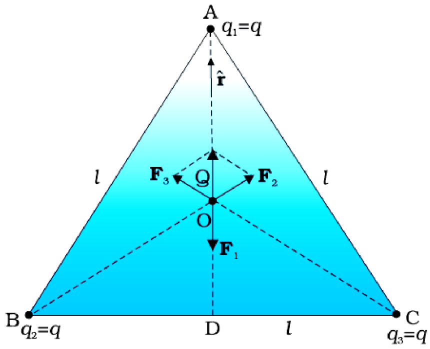
Similarly the total force on charge $-q$ at C is $F_3 = F_3\hat{n}$, where $\hat{n}$ is the unit vector along the direction bisecting the $\angle$BCA. It is interesting to see that the sum of the forces on the three charges is zero, i.e.,

$F_1 + F_2 + F_3 = 0$

The result is not at all surprising. It follows straight from the fact that Coulomb's law is consistent with Newton's third law. The fact that there is an equal and opposite reaction to every action, electrostatic forces between charges are equal and opposite as they are mutual forces.

Example:

Consider three charges q_1, q_2, q_3 each equal to q at the vertices of an equilateral triangle of side l. What is the force on a charge Q (with the same sign as q) placed at the centroid of the triangle?



Solution:

In the given equilateral triangle ABC of sides of length l, if we draw a perpendicular AD to the side BC,

AD = AC cos 30º= (√3 /2) l and the distance AO of the centroid O from A is (2/3) AD = (1/√3) l. By symmetry AO = BO = CO.

Thus,

Force F_1 on Q due to charge q at A: $\frac{3}{4\pi\varepsilon_0}\frac{Qq}{l^2}$

Along AO Force F_2 on Q due to charge q at B: $\frac{3}{4\pi\varepsilon_0}\frac{Qq}{l^2}$

Along BO Force F_3 on Q due to charge q at C: $\frac{3}{4\pi\varepsilon_0}\frac{Qq}{l^2}$

Along CO the resultant of forces F_2 and F_3: $\frac{3}{4\pi\varepsilon_0}\frac{Qq}{l^2}$

Along OA, by the parallelogram law-

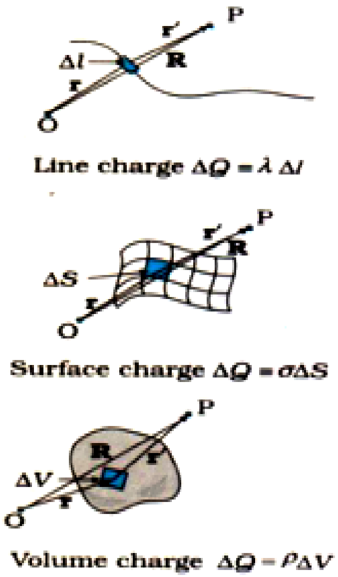
Therefore, the total force on Q: $\frac{3}{4\pi\varepsilon_0}\frac{Qq}{l^2}\left(\hat{r} - \hat{r}\right) = 0$

Where r^ is the unit vector along OA.

It is clear also by symmetry that the three forces will sum to zero. Suppose that the resultant force was non-zero but in some direction.

Continuous Distribution Of Charges

We have so far dealt with point charges or discrete charges q_1, q_2…..q_n. One reason why we have, so far, restricted ourselves to discrete charges is that the mathematical treatment is simpler and does not involve calculus.



However, on a charged body, the amount of charge is enormous as compared to the charge on an electron or a proton (they are discrete charges at microscopic level) and hence we need to talk about continuous charge distributions.

For example, on the surface of a charged conductor, it is impractical to specify the charge distribution in terms of the infinite locations of the microscopic charged constituents. It is more feasible to consider an area elements on the surface of the conductor (which is very small on the macroscopic scale but large enough to include a very large number of electrons) and specify the charge Δq on that element.

We then define a **surface charge density** σ at the area element by $\sigma = \frac{\Delta Q}{\Delta s}$.

We can do this, for different points on the conductor and thus arrive at a continuous function called the surface charge density. The surface charge density so defined ignores the quantization of charge and the discontinuity in charge distribution at the microscopic level. We represent surface charge density at macroscopic level, as average of the microscopic charge density over an area element ΔS.

The SI unit of σ is C/m^2

Similar considerations apply for a **line charge distribution** and a **volume charge distribution.**

The **linear charge density** λ of a wire is defined by $\lambda = \frac{\Delta Q}{\Delta l}$.

where Δl is a small line element of wire on the macroscopic scale that, however, includes a large number of microscopic charged constituents and ΔQ is the charge contained in that line element.

The S.I unit for λ is C/m.

The **volume charge density** (sometimes simply called charge density is defined in a similar manner: $\rho = \frac{\Delta Q}{\Delta V}$ *where* ΔQ is the charge included in the macroscopically small volume element ΔV that includes a large number of microscopic charged constituents.

The S.I unit for ρ is C/m^3.

The notion of continuous charge distribution is similar to that we adopt for continuous mass distribution in mechanics. When we refer to the density of a liquid, we are referring to its macroscopic density. We regard it as a continuous fluid and ignore its discrete molecular constitution.

Thus for continuous charge distribution Coulomb's Law is modified.

The force on a test charge q due to continuous charge distribution can be obtained in much the same way as for a system of discrete charges. First force on test charge is calculated by considering the small charge ΔQ at any arbitrary position placed inside the continuous charge distribution Q. Then total force due to Q is obtained by integrating or taking the summation over the expression.

$$\mathbf{F} = \frac{q}{4\pi\varepsilon_0} \int \frac{\Delta Q}{r^2}$$

Where, for line charge $\Delta Q = \lambda\, dl$,

For, surface charge $\Delta Q = \sigma\, ds$,

For, volume charge $\Delta Q = \rho\, dv$

Summary

Coulomb's law is valid only for point charges. Electric force of interaction between two extended charged bodies is not exactly equal to

$$F = K \frac{q_1 \times q_2}{r^2}$$

Superposition Principle

The principle is based on the property that the forces with which two charges attract or repel each other are not affected by the presence of a third (or more) additional charge(s). For an assembly of charges q_1, q_2, q_3, ..., The force on any charge, say q_1, is the vector sum of the force on q_1 due to q_2, the force on q_1 due to q_3, and so on. For each pair, the force is given by Coulomb's law for two point charges.

Calculation of net force at any point charge due to multiple charges can be done by considering the magnitude and direction of individual forces due to each of the charges and calculating the net force by method of vector addition

It is not necessary that charges exist as point charges, they may be distributed over any shape of body.

Charge distribution along a length of conductor such as a wire will have linear charge density. On a surface will have surface charge density or volume charge density.

This idea is important as objects do not exist as point objects in nature.

The net force due to charge distributions is obtained by integrating or taking the summation over the expression of force for a small element ΔQ.

$$F=\frac{q}{4\pi\varepsilon_0}\int\frac{\Delta Q}{r^2}$$